XVIII.

SURFACES OF THE SECOND ORDER, AS TREATED BY QUATERNIONS.

THE THESIS OF A CANDIDATE FOR MATHEMATICAL HONORS CONFERRED WITH THE DEGREE OF A.B., AT HARVARD COLLEGE, AT COMMENCEMENT, 1877.

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THE surfaces of the second order, or *Quadrics*, as they are very commonly called, present by far too great a field to be investigated in every part in any single thesis. I have therefore chosen only a few branches of the subject; and I have been guided in the selection chiefly by a desire to avoid, as much as possible, those portions of the subject which have been the most thoroughly treated by Hamilton. With this object in view, I have passed over entirely the vast field of *foci* and *confocal surfaces*, and have touched but slightly upon *cyclic normals* and *asymptotic cones*. I have been especially attracted to consider the relations existing between the various conjugate lines and planes of any quadric, and the general relations which the different classes of quadrics bear to each other.

It has also been my object to exhibit that variety of expression which is so peculiar to quaternions, by approaching all questions from more than one point of view. With this idea, I have studied many of the cases arising under the self-conjugate function $\varphi\rho$ under both the *cyclic* and the *rectangular* forms, showing how these forms give different expressions to the same result. And finally, considering it a great advantage to be as general as possible in the treatment of any mathematical subject, I have tried to keep the self-conjugate function under the general form $\varphi\rho$, without attending to the special forms of the terms which compose it.

The equation, $\qquad S\rho\varphi\rho = \text{constant},$

where $\varphi\rho$ is any vector function of ρ, represents in general a surface; for if we write

$$\rho = xi + yj + zk,$$

and if we assume any arbitrary values for x and y, we shall have a scalar equation to determine the corresponding value of z. Our equation, then, represents a surface for the same reason that any one equation between Cartesian co-ordinates represents a surface. It is almost needless to add that, since $\varphi\varrho$ may be any vector function of ϱ, the converse proposition is true: that any surface may be expressed by an equation of the form

$$\mathrm{S}\varrho\varphi\varrho = \text{constant.}$$

The degree of the surface is higher by unity than the degree of the function $\varphi\varrho$, — understanding, as usual, by the degree of a surface the greatest number of times it can be cut by a right line. For suppose that $\varphi\varrho$ is of the n^{th} degree, and we want the intersections of the surface with the line,

$$\varrho = x\alpha.$$

$\varphi\varrho$ may be divided into the sum of vector functions, each of which is homogeneous with regard to ϱ, the highest being of the n^{th} degree. We shall thus have

$$\mathrm{S}\varrho\varphi\varrho = \mathrm{S}\varrho\varphi'\varrho + \mathrm{S}\varrho\varphi''\varrho + \&\text{c.}$$

$$= \mathrm{T}\varrho\mathrm{T}\varphi'\varrho \cos <^{\rho}_{\phi'\rho} + \mathrm{T}\varrho\mathrm{T}\varphi''\varrho \cos <^{\rho}_{\phi''\rho} + \&\text{c.} = c.$$

Now we may substitute $x\alpha$ for ϱ in this equation. But $\cos <^{\rho}_{\phi'\rho}$ depends only on the direction of ϱ and $\varphi'\varrho$, and since we may write

$$\varphi'\varrho = \varphi'(\mathrm{T}\varrho\mathrm{U}\varrho) = \mathrm{T}^n\varrho\varphi'(\mathrm{U}\varrho),$$

the direction of $\varphi'\varrho$ depends only on that of ϱ. Hence, the cosines in the above equation are independent of x.

Now $\mathrm{T}\varphi\varrho$ contains x to the same degree in each term as it does ϱ; that is, to the n^{th} degree in the highest term. Thus the equation

$$\mathrm{T}\varrho\mathrm{T}\varphi'\varrho \cos <^{\rho}_{\phi'\rho} + \mathrm{T}\varrho\mathrm{T}\varphi''\varrho \cos <^{\rho}_{\phi''\rho} + \&\text{c.} = c$$

is an algebraic equation of the degree $(n+1)$, which gives $(n+1)$ solutions for x, or $(n+1)$ distances at which the surface is cut by the line

$$\varrho = x\alpha.$$

A surface of the second order may then be represented by an equation of the form

$$S\rho\varphi\rho = c,$$

where φ is a vector function of ρ of the first order.

The most general form of such a function, $\varphi\rho$, will be a function homogeneous in ρ plus a constant vector γ. But the homogeneous function is equivalent to a self-conjugate function $\varphi_0\rho$, plus a term of the form $V\varepsilon\rho$ (Hamilton's Elements, § 349, (4); Tait, § 174). Now we see that

$$S\rho\varphi\rho = S\rho(\varphi_0\rho + V\varepsilon\rho + 2\gamma) = S\rho\varphi_0\rho + 2\ S\gamma\rho = c,$$

writing in the 2 merely for convenience. That is, the homogeneous part of the function may be taken as self-conjugate. If we can next transform the origin to such a point that γ disappears, the surface will be represented by the equation

$$S\rho\varphi_0\rho = c,$$

and in this case all the variable terms will contain ρ to the second degree, so that satisfied by $+\rho$ the equation will also be satisfied by $-\rho$; *i.e.*, the origin will be at the centre. To find this point, write $\rho + \delta$ for ρ, and (dropping the suffix of $\varphi_0\rho$, but remembering that the function is self-conjugate)

$$S\rho\varphi\rho + 2\ S\rho\varphi\delta + 2\ S\gamma\rho + S\delta\varphi\delta + 2\ S\gamma\delta = c.$$

The terms $2\ S\rho\varphi\delta$ and $2\ S\gamma\rho$ take the place of the old term $2\ S\gamma\rho$; and, in order that they may disappear, we must have

$$S\rho(\varphi\delta + \gamma) = 0;$$

or, since ρ may have any direction,

$$\varphi\delta + \gamma = 0.$$

This is the condition that must be satisfied in order to transform the origin to the centre; and it gives in general a single finite solution for δ. I shall consider later the cases when this solution is indeterminate or infinite, and the corresponding form of the surface.

Suppose now that the equation

$$\varphi\delta + \gamma = 0$$

has been solved, and the centre found. Our equation then assumes the form

$$S\rho\varphi\rho = - S\delta\varphi\delta - 2\, S\gamma\delta + c,$$

and we may write it $\qquad S\rho\varphi\rho = c.$

By this process we have destroyed the three arbitrary constants involved in γ, and left only the six belonging to the self-conjugate $\varphi\rho$ (Ham. Elem., § 358). This is precisely what should happen, for the general equation of the second degree in Cartesian co-ordinates contains nine arbitrary constants, while by taking the centre as origin three of them are lost.

If in the transformed equation

$$S\rho\varphi\rho = c,$$

the constant vanishes, the equation represents a cone, since we may give any value to the tensor of ρ, as the equation is homogeneous. This case also I shall consider later. If the constant term does not vanish, we can divide by it, and get our equation in the more convenient form

$$S\rho\varphi'\rho = 1,$$

c disappearing into the new self-conjugate function $\varphi'\rho$.

If we differentiate

$$S\rho\varphi\rho = 1,$$

we find (Tait, §§ 132, 251 c)

$$S\rho\varphi d\rho + S d\rho\varphi\rho = 0;$$

and since $\varphi\rho$ is self-conjugate,

$$S d\rho\varphi\rho = S\rho\varphi d\rho,$$

or $$S d\rho\varphi\rho = 0.$$

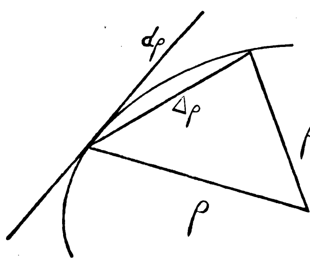

FIG. 1.

But $d\rho$ is in the direction of the variation of ρ at any instant. It is then in the direction of the tangent, at the extremity of ρ (Fig. 1) (Tait, § 36).

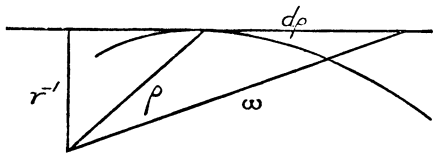

FIG. 2.

Now if we consider ρ fixed, but allow $d\rho$ to vary, we may write $(\omega - \rho)$ for $d\rho$ (Fig. 2), and the equation

$$S d\rho\varphi\rho = S(\omega - \rho)\varphi\rho = 0$$

is that of a plane containing any tangent $d\rho$; and, therefore, of the tangent plane. Since the extremity of ρ is on the surface, we have

$$S\rho\varphi\rho = 1,$$

and the equation of the tangent plane may be written

$$S\omega\varphi\rho = 1.$$

We see (Tait, § 205) that $\varphi\rho$ is perpendicular to this surface, or, in other words, in the direction of the normal; and if we take ω in the direction of $\varphi\rho$, or $= x\varphi\rho$,

$$S x\varphi\rho\varphi\rho = x(\varphi\rho)^2 = 1;$$

$$\therefore\ x\varphi\rho = \frac{1}{\phi\rho} = -\frac{\phi\rho}{T^2\phi\rho};$$

and this last is the perpendicular from the centre on the tangent plane.

CONJUGATE DIAMETERS AND DIAMETRAL PLANES.

If we want to find a line ω through the origin which bisects all chords parallel to another line α, $(\omega + x\alpha)$ and $(\omega - x\alpha)$ must both terminate in the surface: that is, ω must satisfy the equation

$$S(\omega \pm x\alpha)\varphi(\omega \pm x\alpha) = 1,$$

where x is a scalar. Now, if we develop this equation, we find

$$S\omega\varphi\omega + x^2 S\alpha\varphi\alpha \pm 2\,x\,S\omega\varphi\alpha = 1.$$

But this is evidently impossible, unless

$$S\omega\varphi\alpha = 0.$$

The locus of ω is, then, a plane perpendicular to $\varphi\alpha$ and passing through the origin. It is also parallel to

$$S\rho\varphi\alpha = 1,$$

which is the tangent plane at the extremity of a vector through the origin parallel to α. Conversely, if

$$S\omega\varphi\alpha = 0,$$

the locus of ω is a plane bisecting all chords parallel to α, because some x, a scalar function of ω, can evidently be found, such that

$$S(\omega \pm x\alpha)\varphi(\omega \pm x\alpha) = 1.$$

Since φ is self-conjugate, we have

$$S\alpha\varphi\omega = S\omega\varphi\alpha = 0,$$

so that the relation is reciprocal; and if ω be constant, and α vary, the locus of the latter is a plane parallel to the tangent plane at the extremity of a diameter parallel to ω. If β is any vector lying in the first plane, our two planes will be denoted by

$$S\omega\varphi\alpha = 0$$

and

$$S\omega\varphi\beta = 0,$$

and we have

$$S\alpha\varphi\beta = S\beta\varphi\alpha = 0.$$

The intersection of the two planes is $V\varphi\alpha\varphi\beta$, because this satisfies both equations: for

$$S\varphi\alpha\varphi\beta\varphi\alpha = 0 = S\varphi\alpha\varphi\beta\varphi\beta.$$

And, denoting this intersection by γ, we see that

$$S\alpha\varphi\gamma = S\gamma\varphi\alpha = 0,$$

and

$$S\beta\varphi\gamma = S\gamma\varphi\beta = 0.$$

Now

$$S\omega\varphi\gamma = 0$$

is the equation of a plane through the origin perpendicular to $\varphi\gamma$, and bisecting all chords parallel to γ. But such a plane must be that of α and β, for

$$S(x\alpha + y\beta)\varphi\gamma = xS\alpha\varphi\gamma + yS\beta\varphi\gamma = 0.$$

Thus we have a system of three planes, each of which bisects all chords parallel to the intersection of the other two. Hence, if three vectors α, β, γ are such that

$$S\alpha\varphi\beta = 0 = S\beta\varphi\alpha,$$

$$S\alpha\varphi\gamma = 0 = S\gamma\varphi\alpha,$$

$$S\beta\varphi\gamma = 0 = S\gamma\varphi\beta,$$

the diameters parallel to α, β, γ are conjugate diameters, and the planes,

$$S\omega\varphi\alpha = 0,\ S\omega\varphi\beta = 0,\ S\omega\varphi\gamma = 0$$

are conjugate diametral planes. The above equations, which may be written in the form

$$\cos < {}^{\phi\alpha}_{\beta} = 0,\ \cos < {}^{\phi\gamma}_{\alpha} = 0, \text{ and } \cos < {}^{\phi\beta}_{\gamma} = 0,$$

give three conditions for determining the directions of α, β, γ ; since the direction of $\varphi\varrho$ depends only on the direction of ϱ. But three directions involve six arbitrary constants, of which we see that three may be selected arbitrarily. Thus, if one diameter, or one plane, be chosen, the other two can still be taken in an infinity of ways.

Again γ, for instance, bisects all chords through it parallel to the plane of α and β; because, if $\delta = a\alpha + b\beta$,

$$S\gamma\varphi\delta = aS\gamma\varphi\alpha + bS\gamma\varphi\beta = 0.$$

Hence the equation

$$S(m\gamma \pm x\delta)\varphi(m\gamma \pm x\delta)$$

$$= m^2S\gamma\varphi\gamma + x^2S\delta\varphi\delta \pm 2\,mxS\gamma\varphi\delta$$

$$= m^2S\gamma\varphi\gamma + x^2S\delta\varphi\delta = 1,$$

is satisfied by equal and opposite values of x.

PRINCIPAL DIAMETERS.

For any self-conjugate function $\varphi\varrho$, there are three real directions at right angles to each other, and in general only three directions, for which $\varphi\varrho$ is parallel to ϱ (Ham. Elem., § 354). We have already seen that $\varphi\varrho$ has the direction of the normal at the extremity of ϱ. If, then, ϱ is in any one of the three rectangular directions for which $\varphi\varrho$ is parallel to ϱ, its tangent plane must be parallel to the plane of the other two; which must therefore bisect all chords parallel to ϱ. These three directions are, therefore, those of a set of conjugate diameters.

We can see the same thing in a purely analytical way. Let i, j, k represent unit-vectors in the three rectangular directions determined by the above condition; and let $\varphi i = - c_1 i$, $\varphi j = - c_2 j$, $\varphi k = - c_3 k$. Then

$$Si\varphi j = - c_2 Sij = 0,$$

$$Sj\varphi k = - c_3 Sjk = 0,$$

$$Sk\varphi i = - c_1 Ski = 0.$$

Conversely, if α, β, γ are mutually rectangular, they must be respectively parallel to $\varphi\alpha, \varphi\beta, \varphi\gamma$. There is thus one set, and in general only one set, of conjugate diameters which are mutually rectangular.

The reciprocals of the scalar coefficients c_1, c_2, c_3, are the squares of the semiaxes of the quadric; for

$$S\varrho\varphi\varrho = T^2\varrho S\alpha\varphi\alpha,$$

where α is an unit vector in the direction of ϱ. But in the direction of a principal diameter, as i: —

$$T^2\varrho S\alpha\varphi\alpha = - T^2\varrho Sic_1 i = c_1 T^2\varrho = 1.$$

Hence, $$T\varrho = \frac{1}{\sqrt{c_1}},$$

and the semiaxis is $c_1^{-\frac{1}{2}} i$. In the same way, the other semiaxes are $c_2^{-\frac{1}{2}} j$ and $c_3^{-\frac{1}{2}} k$.

If one of the c's is negative, its square root is imaginary, and therefore the radius vector does not cut the surface in that direction, and the quadric is a *single sheeted hyperboloid.* If two c's become negative, only one of the principal axes really cuts the surface, which is a *double sheeted hyperboloid.* If all three c's are positive, the surface is cut in real points by all three axes, and the quadric is an *ellipsoid.* But

if all the c's are negative, there are no real semiaxes, and we get the so-called *imaginary ellipsoid.*

It is well known (Ham. Elem., § 354; Tait, §§ 163, 164) that the three c's are the roots of an algebraic cubic, and are always real. We shall find it convenient to take these roots in the algebraic order: $c_1 < c_2 < c_3$. The general scalar equation,

$$S\rho\varphi\rho = 1,$$

where φ is self-conjugate, may be written in the rectangular form:—

$$c_1 S^2 i\rho + c_2 S^2 j\rho + c_3 S^2 k\rho = 1.$$

If any two roots, as c_2 and c_3, are equal, a plane

$$Si\rho = m$$

perpendicular to the third direction (i) cuts the surface in the circular section,

$$c_2 S^2 j\rho + c_2 S^2 k\rho = 1 - c_1 m^2.$$

So the quadric is a surface of revolution. If all the roots are equal — and this, of course, can only happen in the case of an ellipsoid,—

$$c_1 S^2 i\rho + c_1 S^2 j\rho + c_1 S^2 k\rho = 1,$$

and therefore
$$S^2 i\rho + S^2 j\rho + S^2 k\rho = \frac{1}{c_1},$$

$$\text{or, } T^2\rho\, (\cos^2 \angle \tfrac{i}{\rho} + \cos^2 \angle \tfrac{j}{\rho} + \cos^2 \angle \tfrac{k}{\rho}) = T^2\rho = \frac{1}{c_1},$$

and
$$T\rho = \frac{1}{\sqrt{c_1}}.$$

The surface is then a sphere with $c_1{}^{-\frac{1}{2}}$ for a radius.

If the constant term vanishes, we have already seen that the surface must be a cone. Neither of the principal diameters can be in the direction of a side of the cone; for, if

$$\rho = xi$$

for instance, we find

$$c_1 S^2 ixi = x^2 c_1 = 0;$$

and therefore
$$x = 0.$$

Poles and Polar Planes.

To find the locus of the point of harmonic division of radii through a point α, transfer the origin to that point, and the equation

$$S\varrho\varphi\varrho = 1$$

becomes

$$S\varrho\varphi\varrho + 2\, S\varrho\varphi\alpha + S\alpha\varphi\alpha = 1.$$

Let the vectors of the surface on any line passing through the new origin be ϱ' and ϱ'', and let their harmonic mean be ϱ. We must have then

$$\frac{2}{T\rho} = \frac{1}{T\rho'} + \frac{1}{T\rho''}.$$

If, now, we take ϱ in the direction β, the equation of the surface becomes

$$T\varrho^2 S\beta\varphi\beta + 2T\varrho S\beta\varphi\alpha + S\alpha\varphi\alpha = 1.$$

$T\varrho'$ and $T\varrho''$ are the roots of this quadratic; and applying to it the well-known principles of quadratics, we have

$$\frac{2}{T\rho} = \frac{1}{T\rho'} + \frac{1}{T\rho''} = \frac{T\rho'' + T\rho'}{T\rho' T\rho''} = \frac{-2\, S\beta\phi\alpha}{S\beta\phi\beta} \div \frac{S\alpha\phi\alpha - 1}{S\beta\phi\beta} = \frac{-2\, S\beta\phi\alpha}{S\alpha\phi\alpha - 1};$$

which gives

$$T\varrho S\beta\varphi\alpha = 1 - S\alpha\varphi\alpha;$$

or, since β is the versor of ϱ,

$$S\varrho\varphi\alpha + S\alpha\varphi\alpha = 1;$$

which is the equation of a plane, the polar plane of the new origin. Transfer back to the former origin, by substituting $\varrho - \alpha$ for ϱ, and we find

$$S\varrho\varphi\alpha = 1$$

as the equation of the polar plane of α.

This last equation can be written

$$S\alpha\varphi\varrho = 1,$$

and by changing the variable we get, of course, the polar plane of ϱ. So we see that, if α is on the polar plane of β ($S\alpha\varphi\beta = 1$), β is on the polar plane of α ($S\beta\varphi\alpha = 1$).

We have found that the equation of the tangent plane at a point ϱ is

$$S\omega\varphi\varrho = 1.$$

If we consider ω fixed and ρ variable, we shall find the equation of a plane containing all the points where tangents from ω meet the surface. But this

$$S\rho\varphi\omega = 1$$

is the same equation that we have just found for the polar plane of ω. And this is indeed what we should expect, for if two radii vectores from ω become equal, their harmonic mean is equal to each of them, and must reach the surface where they do. We see that the polar plane of any point on a tangent plane must pass through the point of contact, and that the polar plane of any point cuts the surface in the locus of the points of contact of tangent lines drawn from that point.

Relations between Polar Planes and Conjugate Diameters.

The function $\varphi\rho$ is, as we have seen, not changed in direction by varying the tensor of ρ. The polar planes, then, of all points in the same straight line from the origin are parallel, for they are all represented by

$$S\rho\varphi\alpha = 1,$$

where the tensor only of α varies. But the polar plane

$$S\rho\varphi\alpha = 1,$$

where α is a vector of the surface, becomes a tangent plane; and this is parallel to

$$S\rho\varphi\alpha = 0,$$

the diametral plane bisecting all chords parallel to α. Hence, we find that the polar plane of any point is parallel to the diametral plane conjugate to the diameter passing through the point. From another property of conjugate diameters, it is seen that the diameter through any point bisects all chords that it meets parallel to the polar plane of that point. Again,

$$S\alpha\varphi\rho = 1$$

is the same as

$$S\alpha(\varphi\rho - \alpha^{-1}) = 0;$$

and, in order that this should be equal to

$$S\alpha\varphi\rho = 0,$$

α^{-1} must vanish and α become infinite. Thus, in the same way that a tangent plane is the polar plane of a point on the quadric, a diametral plane is the polar plane of a point at infinity.

From the relation between polar planes and conjugate diameters, it is evident that there are three rectangular directions for which the polar plane of a point is perpendicular to the vector from the origin to that point. It is, perhaps, needless to add that in cases where two of the roots c_1, c_2, c_3 are equal, — that is, where the directions for which $\varphi\rho$ is parallel to ρ, degenerate into one vector and any two in a plane perpendicular to it, — the directions for which central radii and polar planes are perpendicular degenerate in the same way, and we have surfaces of revolution.

In the central equation of the cone, we have already noticed that the constant term vanishes. If, now, we take the general central equation of the cone in the form

$$S\rho\varphi\rho = 0,$$

a tangent plane to the surface at any point α is represented by

$$S\rho\varphi\alpha = 0;$$

but this is satisfied by

$$\rho = 0.$$

Every tangent plane to a cone, then, passes through the centre, or vertex. The equation of the polar plane of any point is

$$S\rho\varphi\alpha = 0,$$

and this also passes through the centre. It is parallel to the diametral plane conjugate to α, and since both of these planes pass through the centre they must coincide. Indeed, both are represented by the same equation

$$S\rho\varphi\alpha = 0.$$

We see, moreover, that this is the polar plane of all points on the line

$$\rho = x\alpha,$$

because all such planes are parallel and all pass through the origin.

Cyclic Normals.

I wish to say only a word about these remarkable vectors, in order to show the connection between the equation of the *central quadrics* and the self-conjugate part of that of the *paraboloids*. If we take the cyclic transformation of the equation

$$c = S\rho\phi\rho = g\rho^2 + S\rho V\lambda\rho\mu$$
$$= g\rho^2 + S\rho(-\rho S\lambda\mu + \lambda S\mu\rho + \mu S\lambda\rho)$$
$$= \rho^2(g - S\lambda\mu) + 2S\lambda\rho S\mu\rho\,;$$

and if we cut the surface by planes perpendicular to λ or μ, — *i.e.*, by the planes

$$S\lambda\rho = c',$$

$$S\mu\rho = c'',$$

we find

$$(g - S\lambda\mu)\rho^2 + 2\,c' S\mu\rho = c,$$

$$(g - S\lambda\mu)\rho^2 + 2\,c'' S\lambda\rho = c,$$

either of which is a sphere whose intersection with the plane is a circle. This would still be true if c should vanish, and the quadric become a cone; the only difference being that in this case the origin would be on the surface of the sphere.

Tangent Cone.

The plane passing through all the points of contact of tangents from α to the quadric

$$S\rho\phi\rho = 1,$$

is the polar plane of α, as we have seen, and its equation is

$$S\rho\phi\alpha = 1.$$

A surface of the second order tangent to the quadric along its intersection with this plane will be represented by

$$S\rho\phi\rho - 1 + x(S\rho\phi\alpha - 1)^2 = 0.$$

If this surface pass through α, its equation must be satisfied by α, and therefore

$$S\alpha\phi\alpha - 1 + x(S\alpha\phi\alpha - 1)^2 = 0,$$

and this gives

$$x = \frac{-1}{(S\alpha\phi\alpha - 1)}.$$

Substituting this value of x, we obtain the equation

$$(S\rho\phi\rho - 1)\,(S\alpha\phi\alpha - 1) - (S\rho\phi\alpha - 1)^2 = 0.$$

Transfer the origin to α, and this becomes

$$(S\varrho\varphi\varrho + 2\,S\varrho\varphi\alpha + S\alpha\varphi\alpha - 1)(S\alpha\varphi\alpha - 1) - (S\varrho\varphi\alpha + S\alpha\varphi\alpha - 1)^2$$
$$= S\varrho\varphi\varrho(S\alpha\varphi\alpha - 1) - (S\varrho\varphi\alpha)^2 = 0.$$

This equation represents a cone referred to its centre, because every term contains ϱ to the second degree. It is, then, the *tangent cone* from α.

If in the equation of the polar plane

$$S\varrho\varphi\alpha = 1$$

α vanish, no finite value of ϱ will satisfy the equation, and the polar plane of the origin is seen to lie at infinity. The tangent cone from the origin must therefore meet the quadric at infinity, and becomes what is called the *asymptotic cone.* The equation of this cone is readily obtained by substituting

$$\alpha = 0$$

in that of the tangent cone

$$S\varrho\varphi\varrho(S\alpha\varphi\alpha - 1) - (S\varrho\varphi\alpha)^2 = 0,$$

which gives

$$S\varrho\varphi\varrho = 0.$$

The rectangular transformation for this is

$$c_1 S^2\alpha_1\varrho + c_2 S^2\alpha_2\varrho + c_3 S^2\alpha_3\varrho = 0,$$

and it can only be satisfied by real finite values of ϱ, where one or more c's are negative, and the quadric an *hyperboloid.* In the case of the real or imaginary *ellipsoid,* the c's are all of the same sign, and the asymptotic cone is reduced to its vertex, for its equation is only satisfied by

$$\varrho = 0.$$

The reality of any cone depends of course, in the same way, on the difference of the signs of the roots of $\varphi\varrho$. Any cone may, then, be regarded as the limiting case of an hyperboloid which is degenerating into its own asymptotic cone.

This idea leads us to consider how one form of surface of the second order may, by the modification of the constants in its equation, pass by imperceptible degrees into some different form.

When any root of a central quadric vanishes, the surface becomes indeterminate in the direction of that root, and thus degenerates into a cylinder. If c_1, for instance, vanishes, we have

$$c_2 S^2\alpha_2\varrho + c_3 S^2\alpha_3\varrho = 1,$$

and this represents a *cylinder;* because to any radius vector ρ we can add $x\alpha_1$, where x is any scalar. But what is to become of the asymptotic cone in this case? It must also be indeterminate in the same direction, and yet it must still retain the property that all radii vectores must lie wholly in its surface. The only surfaces of the second order of which this can be true are pairs of real or imaginary planes.

The quaternion expression for this is very interesting. If the negative root of a single-sheeted hyperboloid or either root of an ellipsoid vanishes, the quadric is represented by the equation

$$c_2 S^2 \alpha_2 \rho + c_3 S^2 \alpha_3 \rho = 1,$$

and becomes an elliptic cylinder. The asymptotic cone

$$c_2 S^2 \alpha_2 \rho + c_3 S^2 \alpha_3 \rho = 0,$$

or

$$c_2 S^2 \alpha_2 \rho = - c_3 S^2 \alpha_3 \rho,$$

or

$$\sqrt{c_2} S \alpha_2 \rho = \pm \sqrt{- c_3} S \alpha_3 \rho,$$

becomes a pair of imaginary planes, containing only one real line,

$$\rho = x\alpha_1,$$

the line of their intersection, which satisfies the equation because it makes both sides vanish. This may be considered a sort of intermediate case, on a roundabout road, between the real (hyperbolic) asymptotic cone and the imaginary (elliptic) one.

If the positive root of a double-sheeted hyperboloid vanish, the surface

$$- c_1 S^2 \alpha_1 \rho - c_2 S^2 \alpha_2 \rho = 1$$

becomes an imaginary elliptic cylinder. The asymptotic cone

$$- c_1 S^2 \alpha_1 \rho - c_2 S^2 \alpha_2 \rho = 0,$$

or

$$\sqrt{- c_1} S \alpha_1 \rho = \pm \sqrt{c_2} S \alpha_2 \rho,$$

again represents a pair of imaginary planes, containing only one real line,

$$\rho = x\alpha_3,$$

this time at right angles to the former direction.

If, finally, a positive root of a single-sheeted hyperboloid or a nega-

tive root of a double-sheeted one vanish, so as to leave the two actual roots with opposite signs, the quadric degenerates into

$$c_1 S^2 \alpha_1 \rho - c_3 S^2 \alpha_3 \rho = 1,$$

an hyperbolic cylinder. In this case, the asymptotic cone is

$$c_1 S^2 \alpha_1 \rho - c_3 S^2 \alpha_3 \rho = 0,$$

$$c_1 S^2 \alpha_1 \rho = c_3 S^2 \alpha_3 \rho,$$

or

$$\sqrt{c_1} S \alpha_1 \rho = \pm \sqrt{c_3} S \alpha_3 \rho,$$

a pair of real planes tangent to the cylinder at infinity.

When two roots vanish,

$$c_3 S^2 \alpha_3 \rho = 1$$

represents two parallel planes, real or imaginary, according as the actual root is positive or negative. The asymptotic cone

$$c_3 S^2 \alpha_3 \rho = 0,$$

or

$$S \alpha_3 \rho = 0,$$

is a plane — which we might call a double plane — passing through the origin and parallel to the pair of planes. These cases of degeneracy of quadrics we are about to study from an entirely different point of view.

Non-Central Quadrics.

It has been already proved (page 224) that the centre of a quadric is found by solving for δ the equation

$$\varphi_0 \delta + \gamma = 0.$$

Now the self-conjugate function may be treated under several different forms, such as the *rectangular*, *cyclic*, or *focal* (Ham. Elem., § 359, I., III., and V.). Of these, I shall, for the sake of generality, consider two, the *rectangular* and the *cyclic*, although the former is far more convenient than the latter.

To solve the equation

$$\varphi_0 \delta + \gamma = 0,$$

I shall use the general formula (Ham., §§ 347–350; Tait, Chap. V.)

$$m\rho = m\varphi^{-1}\gamma = m'\gamma - m''\varphi\gamma + \varphi^2\gamma,$$

where

$$m = \frac{S\phi'\lambda\phi'\mu\phi'\nu}{S\lambda\mu\nu},$$

$$m' = \frac{S(\lambda\phi'\mu\phi'\nu + \phi'\lambda.\mu.\phi'\nu + \phi'\lambda\phi'\mu.\nu)}{S\lambda\mu\nu},$$

and

$$m'' = \frac{S(\lambda\mu\phi'\nu + \lambda\phi'\mu.\nu + \phi'\lambda.\mu.\nu)}{S\lambda\mu\nu}.$$

Since the function we are considering is self-conjugate,

$$\varphi_0 = \varphi_0'.$$

Let us take first the *rectangular* form

$$\varphi_0\delta = c_1\alpha_1 S\alpha_1\delta + c_2\alpha_2 S\alpha_2\delta + c_3\alpha_3 S\alpha_3\delta = -\gamma;$$

and let

$$\alpha_1 = \lambda, \ \alpha_2 = \mu, \text{ and } \alpha_3 = \nu,$$

then

$$\varphi\lambda = \varphi\alpha_1 = -c_1\alpha_1,$$

$$\varphi\mu = \varphi\alpha_2 = -c_2\alpha_2,$$

$$\varphi\nu = \varphi\alpha_3 = -c_3\alpha_3.$$

Substituting these values in our formulas,

$$m = \frac{-c_1c_2c_3 S\alpha_1\alpha_2\alpha_3}{S\alpha_1\alpha_2\alpha_3} = -c_1c_2c_3,$$

$$m' = \frac{S(\alpha_1.c_2\alpha_2.c_3\alpha_3 + c_1\alpha_1.\alpha_2.c_3\alpha_3 + c_1\alpha_1.c_2\alpha_2.\alpha_3)}{S\alpha_1\alpha_2\alpha_3} = c_2c_3 + c_1c_3 + c_1c_2,$$

$$m'' = \frac{S(-\alpha_1.\alpha_2.c_3\alpha_3 - \alpha_1.c_2\alpha_2.\alpha_3 - c_1\alpha_1.\alpha_2.\alpha_3)}{S\alpha_1\alpha_2\alpha_3} = -c_3 - c_2 - c_1.$$

And we find for the general equation of solution

$$\begin{aligned} m\delta = -c_1c_2c_3\delta = &-(c_1c_2 + c_1c_3 + c_2c_3)\gamma - (c_1 + c_2 + c_3)(c_1\alpha_1 S\alpha_1\gamma \\ &+ c_2\alpha_2 S\alpha_2\gamma + c_3\alpha_3 S\alpha_3\gamma) + c_1^2\alpha_1 S\alpha_1\gamma + c_2^2\alpha_2 S\alpha_2\gamma + c_3^2\alpha_3 S\alpha_3\gamma \\ = &-(c_1c_2 + c_1c_3 + c_2c_3)\gamma - (c_1c_2 + c_1c_3)\alpha_1 S\alpha_1\gamma \\ &-(c_2c_3 + c_2c_1)\alpha_2 S\alpha_2\gamma - (c_3c_1 + c_3c_2)\alpha_3 S\alpha_3\gamma; \end{aligned}$$

and for the complete solution

$$\begin{aligned} \delta = &+\left(\frac{1}{c_3} + \frac{1}{c_2} + \frac{1}{c_1}\right)\gamma + \left(\frac{1}{c_3} + \frac{1}{c_2}\right)\alpha_1 S\alpha_1\gamma + \left(\frac{1}{c_1} + \frac{1}{c_3}\right)\alpha_2 S\alpha_2\gamma \\ &+ \left(\frac{1}{c_2} + \frac{1}{c_1}\right)\alpha_3 S\alpha_3\gamma. \end{aligned}$$

This will evidently give a single finite value for δ, unless one of the c's vanish. Take the most general case. Let γ be in any direction, such as

$$\gamma = - d\alpha_1 - g\alpha_2 - k\alpha_3.$$

Then

$$S\alpha_1\gamma = - dS\alpha_1^2 = + d,$$

$$S\alpha_2\gamma = + g, \text{ and } S\alpha_3\gamma = + k,$$

$$\delta = \left(\frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3}\right)(- d\alpha_1 - g\alpha_2 - k\alpha_3) + \left(\frac{1}{c_2} + \frac{1}{c_3}\right) d\alpha_1 + \left(\frac{1}{c_3} + \frac{1}{c_1}\right) g\alpha_2 + \left(\frac{1}{c_1} + \frac{1}{c_2}\right) k\alpha_3$$

$$= - \frac{1}{c_1} d\alpha_1 - \frac{1}{c_2} g\alpha_2 - \frac{1}{c_3} k\alpha_3.$$

Then if one root, say c_1, vanishes, the centre is at an infinite distance in the direction of α_1, but at a finite distance in each of the other directions. Now let us substitute

$$c_1 = 0,$$

and

$$\gamma = - d\alpha_1 - g\alpha_2 - k\alpha_3$$

in the general equation of the second degree

$$S\rho\varphi\rho + 2\, S\gamma\rho = c = c_1 S^2\alpha_1\rho + c_2 S^2\alpha_2\rho + c_3 S^2\alpha_3\rho + 2\, S\gamma\rho.$$

We find

$$c_2 S^2\alpha_2\rho + c_3 S^2\alpha_3\rho - 2\, dS\alpha_1\rho - 2\, gS\alpha_2\rho - 2\, kS\alpha_3\rho = c.$$

We found that $\frac{1}{c_2}g$ and $\frac{1}{c_3}k$ were the distances of the centre in the directions of α_2 and α_3. To bring the origin into a line with the centre in these two directions, substitute

$$\rho = \rho - \frac{g}{c_2} \alpha_2 - \frac{k}{c_3} \alpha_3,$$

and the equation takes the form

$$c_2 S^2\alpha_3\rho + c_3 S^2\alpha_3\rho - 2\, dS\alpha_1\rho = c.$$

This may be still farther simplified, by taking for origin that point where α_1 meets the surface. Let

$$\rho = x\alpha_1,$$

$$-2\,d\mathrm{S}\alpha_1 x\alpha_1 = +x\,2\,d = c,$$

$$x = \frac{+c}{2\,d}.$$

Now substitute

$$\varrho = \varrho + \frac{c}{2\,d}\,\alpha_1,$$

and we find

$$c_2\mathrm{S}^2\alpha_2\varrho + c_3\mathrm{S}^2\alpha_3\varrho - 2\,d\mathrm{S}\alpha_1\varrho + 2\,d\,\frac{c}{2\,d} = c,$$

$$c_2\mathrm{S}^2\alpha_2\varrho + c_3\mathrm{S}^2\alpha_3\varrho - 2\,d\mathrm{S}\alpha_1\varrho = 0.$$

The quadric represented by this equation is an *elliptic* or *hyperbolic paraboloid*, according as the c's are alike or unlike in sign. Because if the surface be cut by a plane

$$\mathrm{S}\alpha_2\varrho = c \text{ or } \mathrm{S}\alpha_3\varrho = c$$

perpendicular to either α_2 or α_3, the section is

$$c_2\mathrm{S}^2\alpha_2\varrho = 2\,d\mathrm{S}\alpha_1\varrho - c,$$

or

$$c_3\mathrm{S}^2\alpha_3\varrho = 2\,d\mathrm{S}\alpha_1\varrho - c,$$

either of which is easily seen to be a *parabola*. But if cut by a plane perpendicular to α_1

$$\mathrm{S}\alpha_1\varrho = c,$$

the section is

$$c_2\mathrm{S}^2\alpha_2\varrho + c_3\mathrm{S}^2\alpha_3\varrho = -c,$$

which is an *ellipse* or *hyperbola*, according as c_2 and c_3 are alike or unlike in sign. The sign of c shows the side of the origin in which the cutting plane lies, and determines whether the elliptic section is real or imaginary, or whether the hyperbolic section has its transverse axis parallel to α_2 or to α_3.

We have considered the case in which

$$c_1 = 0,$$

and

$$\gamma = -d\alpha_1 - g\alpha_2 - k\alpha_3.$$

Let us now go a step farther, and suppose d, g, or k to vanish, and let us first take it as the one corresponding to the root that has disappeared. In this case

$$d = 0;$$

and therefore

$$\gamma = -g\alpha_2 - k\alpha_3.$$

The vector of the centre has been found to be

$$\delta = -\frac{d}{c_1}\alpha_1 - \frac{g}{c_2}\alpha_2 - \frac{k}{c_3}\alpha_3.$$

But, if both d and c_1 vanish, the centre must be at a determinate finite distance in two directions, and at an indeterminate distance in the other. The general equation

$$S\rho\varphi_0\rho + 2\,S\gamma\rho = c$$

becomes

$$c_2 S^2\alpha_2\rho + c_3 S^2\alpha_3\rho - 2\,gS\alpha_2\rho - 2\,kS\alpha_3\rho = c.$$

Reduce this to some point in the central line, by substituting

$$\rho = \rho - \frac{g}{c_2}\alpha_2 - \frac{k}{c_3}\alpha_3,$$

and our equation takes the form

$$c_2 S^2\alpha_2\rho + c_3 S^2\alpha_3\rho = c.$$

This represents a cylinder, because we may add $x\alpha_1$ to any vector without affecting the equation; as, indeed, we can see by inspection of the unreduced form of equation for this case. We have thus found again, by a totally different process, the same case of degeneracy considered on pages 235, 236, and 237. And in comparing our results we must remember that the function there called $\varphi\rho$, and the one called $\varphi_0\rho$ here, are exactly the same; at least, with the exception that the former was multiplied by a constant, in order to make the constant term of the equation equal to unity.

If, in the equation for the cylinder,

$$c_2 S^2\alpha_2\rho + c_3 S^2\alpha_3\rho - 2\,gS\alpha_2\rho - 2\,kS\alpha_3\rho = c,$$

$$c = -\frac{g^2}{c_2} - \frac{k^2}{c_3},$$

the equation is a complete square, and equivalent to

$$\left(\sqrt{c_2}S\alpha_2\rho - \frac{g}{\sqrt{c_2}}\right) = \pm\left(\sqrt{c_3}S\alpha_3\rho - \frac{k}{\sqrt{c_3}}\right).$$

And this represents a pair of real planes, or a pair of imaginary planes with the real intersection

$$\rho = x\alpha_1,$$

according as c_2 and c_3 have like or opposite signs. But these conditions for the value of c are really the same that we found before, in

order that a quadric should degenerate into its asymptotic cone, and this is because if with this value of c we transfer the origin to the centre,

$$\delta = -\frac{g}{c_2}\alpha_2 - \frac{k}{c_3}\alpha_3,$$

we obtain an equation without any absolute term. If in the equation

$$\gamma = -d\alpha_1 - g\alpha_2 - k\alpha_3$$

g or k vanish, when c_1 alone of the roots disappears, it would merely indicate that the origin was already in a line with the centre as far as that direction is concerned.

Suppose two roots, as c_1 and c_2, vanish, and also the co-efficient of γ in the direction of one of them, say d. The solution for the centre will then give

$$\delta = -\frac{1}{c_1}d\alpha_1 - \frac{1}{c_2}g\alpha_2 - \frac{1}{c_3}k\alpha_3.$$

The centre is thus at an infinite distance in one direction, and at an indeterminate distance in another. This can only be true of a *parabolic cylinder.* Let us see what the general equation of the second degree will give us in this case.

$$c_1 S^2\alpha_1\rho + c_2 S^2\alpha_2\rho + c_3 S^2\alpha_3\rho + 2\,S\gamma\rho = c$$

becomes

$$c_3 S^2\alpha_3\rho - 2\,gS\alpha_2\rho - 2\,kS\alpha_3\rho = c.$$

And by cutting this by planes perpendicular to α_1, α_2, and α_3, it can at once be proved to be a *parabolic cylinder.* By substituting

$$\rho = \rho - \frac{k}{c_3}\alpha_3 + \frac{c + \frac{2\,k^2}{c_3}}{2\,g}\alpha_2,$$

the equation can be reduced to its simplest form

$$c_3 S^2\alpha_3\rho - 2\,gS\alpha_2\rho = 0.$$

If, as a farther condition, we had

$$g = 0,$$

this last transformation would not be possible; but the equation could be reduced to the form

$$c_3 S^2\alpha_3\rho = c,$$

or

$$\sqrt{c_3}S\alpha_3\rho = \pm\sqrt{c},$$

and would represent a *pair of planes* perpendicular to α_3, and hence parallel to each other.

For the sake of generality, I shall solve the equation for the centre, and examine a couple of cases arising under it, by means of the *cyclic* transformation. And although the result is far from satisfactory, yet the solution contains some points of interest, and is worth inserting in spite of its length. The well-known formula for this form of transformation is (Ham. Elem., § 357 (5)and (8)):

$$\varphi_0\varrho = g\varrho + \mathrm{V}\lambda\varrho\mu = (g - \mathrm{S}\lambda\mu)\varrho + \lambda\mathrm{S}\mu\varrho + \mu\mathrm{S}\lambda\varrho,$$

where it is found that if we take $c_1 < c_2 < c_3$,

$$c_2 = -g + \mathrm{S}\lambda\mu,$$

$$c_1 = -g - \mathrm{T}\lambda\mu,$$

$$c_3 = -g + \mathrm{T}\lambda\mu,$$

$$a_1 = \mathrm{U}(\lambda\mathrm{T}\mu - \mu\mathrm{T}\lambda),$$

$$a_2 = \mathrm{UV}\lambda u,$$

$$a_3 = \mathrm{U}(\lambda\mathrm{T}\mu + \mu\mathrm{T}\lambda).$$

To solve

$$g\varrho + \mathrm{V}\lambda\varrho\mu = \gamma$$

(changing for the sake of convenience the sign of γ) by means of the formulas of page 238. Let

$$\lambda = \lambda,\ \mu = \mu, \text{ and } \nu = \gamma.$$

Here, again, φ is, of course, self-conjugate as before. Now we find

$$\varphi\lambda = g\lambda + \lambda^2\mu,$$

$$\varphi\mu = g\mu + \lambda\mu^2,$$

$$\varphi\nu = g\gamma + \mathrm{V}\lambda\gamma\mu.$$

And substituting these values: —

$$\begin{aligned}\mathrm{S}\varphi\lambda\varphi\mu\varphi\nu &= \mathrm{S}(g\lambda + \lambda^2\mu)(g\mu + \lambda\mu^2)(g\gamma + \lambda\mathrm{S}\gamma\mu - \gamma\mathrm{S}\lambda\mu + \mu\mathrm{S}\lambda\gamma)\\ &= \mathrm{S}(g^2\lambda\mu + 2\,g\lambda^2\mu^2 + \mu^3\lambda^3)(g\gamma + \lambda\mathrm{S}\gamma\mu - \gamma\mathrm{S}\lambda\mu + \mu\mathrm{S}\lambda\gamma)\\ &= \mathrm{S}(g^3\lambda\mu\gamma + 2\,g^2\lambda^2\mu^2\gamma - g\lambda^3\mu^3\gamma + g^2\lambda^2\mu\mathrm{S}\gamma\mu + 2\,g\lambda^3\mu^2\mathrm{S}\gamma\mu\\ &\quad + \lambda^4\mu^3\mathrm{S}\gamma\mu - g^2\lambda\mu\gamma\mathrm{S}\lambda\mu - 2\,g\lambda^2\mu^2\gamma\mathrm{S}\lambda\mu - \mu^3\lambda^3\gamma\mathrm{S}\lambda\mu + \text{vector terms})\\ &= \mathrm{S}(g^3.\lambda\mu\gamma - g\lambda^2\mu^2.\lambda\mu\gamma - g^2\lambda\mu\gamma.\mathrm{S}\lambda\mu + \lambda\mu\gamma.\lambda^2\mu^2\mathrm{S}\lambda\mu).\end{aligned}$$

From which

$$m = g^3 - g\lambda^2\mu^2 - g^2 S\lambda\mu + \lambda^2\mu^2 S\lambda\mu,$$

$$= (g - S\lambda\mu)(g^2 - \lambda^2\mu^2).$$

To find m'

$$S(\lambda\varphi\mu\varphi\nu) = S(\lambda.(g\mu + \mu^2\lambda)(g\gamma + \lambda S\gamma\mu - \gamma S\lambda\mu + \mu S\lambda\gamma))$$

$$= S(\lambda.(g^2\mu\gamma + g\mu\lambda.S\gamma\mu - g\mu\gamma S\lambda\mu + g\mu^2 S\lambda\gamma + g\mu^2\lambda\gamma + \mu^2\lambda^2 S\lambda\mu - \mu^2\lambda\gamma S\lambda\mu + \lambda\mu^3 S\lambda\gamma))$$

$$= S(g^2\lambda\mu\gamma - gS\lambda\mu.\lambda\mu\gamma)$$

$$= g^2 S\lambda\mu\gamma - gS\lambda\mu.S\lambda\mu\gamma.$$

In the same way

$$S(\varphi\lambda.\mu.\varphi\nu) = g^2 S\lambda\mu\gamma - gS\lambda\mu.S\lambda\mu\gamma.$$

But

$$S(\varphi\lambda.\varphi\mu.\nu) = S(g\lambda + \lambda^2\mu)(g\mu + \mu^2\lambda)\gamma$$

$$= S(g^2\lambda\mu + 2\,g\lambda^2\mu^2 + \mu^3\lambda^3)\gamma$$

$$= g^2 S\lambda\mu\gamma - \lambda^2\mu^2 S\lambda\mu\gamma.$$

Hence

$$m' = 2\,g^2 - 2\,gS\lambda\mu + g^2 - \lambda^2\mu^2 = 3\,g^2 + 2\,gS\lambda\mu - \lambda^2\mu^2.$$

To find m'': —

$$S\varphi\lambda.\mu.\nu = S(g\lambda + \lambda^2\mu)\mu.\gamma = gS\lambda\mu\gamma$$

$$S\lambda.\varphi\mu.\nu = S\lambda(g\mu + \mu^2\lambda)\gamma = gS\lambda\mu\gamma.$$

But

$$S\lambda.\mu.\varphi\nu = S(\lambda.\mu.(g\gamma + \lambda S\mu\gamma - \gamma S\lambda\mu + \mu S\lambda\gamma))$$

$$= S(g\lambda\mu\gamma - S\lambda\mu.\lambda\mu\gamma) = (g - S\lambda\mu)S\lambda\mu\gamma.$$

And therefore,

$$m'' = g + g + g - S\lambda\mu = 3\,g - S\lambda\mu.$$

To find $\varphi^2\gamma$,

$$\varphi\gamma = g\gamma + \lambda S\mu\gamma - \gamma S\lambda\mu + \mu S\lambda\gamma = g\gamma + V\lambda\gamma\mu;$$

hence

$$\varphi^2\gamma = \varphi\varphi\gamma = g(g\gamma + V\lambda\gamma\mu) + V(\lambda(g\gamma + V\lambda\gamma\mu)\mu),$$

$$= g^2\gamma + 2gV\lambda\gamma\mu + V(\lambda V\lambda\gamma\mu.\mu)$$

$$= g^2\gamma + 2\,g(\lambda S\mu\gamma - \gamma S\lambda\mu + \mu S\lambda\gamma) + \mu\lambda^2 S\mu\gamma + \lambda\mu^2 S\lambda\gamma - \lambda S\lambda\mu S\mu\gamma + \gamma S^2\lambda\mu - \mu S\lambda\mu S\lambda\gamma.$$

To find

$$m''\varphi\gamma = (3g - S\lambda\mu)(g\gamma + \lambda S\mu\gamma - \gamma S\lambda\mu + \mu S\lambda\gamma),$$
$$= 3\,g^2\gamma + 3\,g\lambda S\mu\gamma - 3\,g\gamma S\lambda\mu + 3\,g\mu S\lambda\gamma - g\gamma S\lambda\mu$$
$$- \lambda S\lambda\mu S\mu\gamma + \gamma S^2\lambda\mu - \mu S\lambda\mu S\lambda\gamma.$$

We have then

$$m\varphi^{-1}\gamma = m\delta = m'\gamma - m''\varphi\gamma + \varphi^2\gamma = 3\,g^2\gamma - 2\,g\gamma S\lambda\mu - \gamma\lambda^2\mu^2 -$$
$$3\,g^2\gamma - 3\,g\lambda S\mu\gamma + 4\,g\gamma S\lambda\mu - 3\,g\mu S\lambda\gamma + g^2\gamma + 2\,g\lambda S\mu\gamma$$
$$- 2\,g\gamma S\lambda\mu + 2\,g\mu S\lambda\gamma + \mu\lambda^2 S\mu\gamma + \lambda\mu^2 S\lambda\gamma.$$
$$= - \gamma\lambda^2\mu^2 - g\lambda S\mu\gamma - g\mu S\lambda\gamma + g^2\gamma + \mu\lambda^2 S\mu\gamma + \lambda\mu^2 S\lambda\gamma$$
$$= \lambda(\mu^2 S\lambda\gamma - gS\mu\gamma) + \mu(\lambda^2 S\mu\gamma - gS\lambda\gamma) + \gamma(g^2 - \lambda^2\mu^2).$$

And the complete solution for the centre is

$$\delta = \frac{\lambda(\mu^2 S\lambda\gamma - gS\mu\gamma) + \mu(\lambda^2 S\mu\gamma - gS\lambda\gamma) + \lambda(g - T\lambda\mu)(g + T\lambda\mu)}{(g - S\lambda\mu)(g^2 - \lambda^2\mu^2) = (g - S\lambda\mu)(g - T\lambda\mu)(g + T\lambda\mu)}.$$

From the values of c_1, c_2, and c_3 given on page 243, we see that, as we found before under the *rectangular* form, δ has a single finite value, unless one of the roots c_1, c_2, or c_3 of $\varphi_0\varrho$ vanishes. This *cyclic* solution for δ, because it contains no explicit rectangular vectors, is difficult to use; and, indeed, often assumes a hopelessly indeterminate form.

We can, however, obtain with sufficient ease in this cyclic form the equations of the *paraboloids*. It was found under the rectangular transformation that, in order that the equation of the second degree should represent a paraboloid, we must have c_2 or c_1 disappear; that is,

$$g = S\lambda\mu \text{ or } g = - T\lambda\mu.$$

Let us consider the latter case. We know (Ham., § 357 (9), XXII.)

$$S\lambda\varrho\mu\varrho - \varrho^2 T\lambda\mu = \{(S\lambda\mu\varrho)^2 + (S\lambda\varrho.T\mu + S\mu\varrho T\lambda)^2\} \times (T\lambda\mu - S\lambda\mu)^{-1} = 2\,S\gamma\varrho + c,$$

if $$g = - T\lambda\mu;$$

for $$g\varrho^2 + S\lambda\varrho\mu\varrho = 2\,S\gamma\varrho + c$$

is a form of the general equation of the second degree, and therefore

$$(S\lambda\mu\varrho)^2 + (S\lambda\varrho T\mu + S\mu\varrho T\lambda)^2 = (T\lambda\mu - S\lambda\mu)(2\,S\gamma\varrho + c).$$

It is evident from the forms of the terms of the first member that the resolved parts of γ in the rectangular directions $UV\lambda\mu$ and $U(\lambda T\mu + \mu T\lambda)$ can be eliminated by taking a new origin; and the γ reduced to the third direction $U(\lambda T\mu - \mu T\lambda)$.

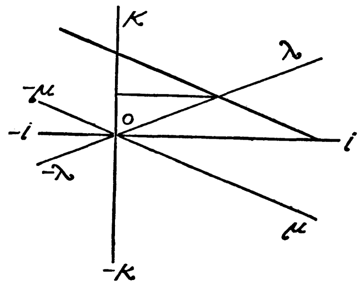

FIG. 3.

Now $$T(\lambda T\mu - \mu T\lambda) = T\lambda\mu T(U\lambda - U\mu)$$

$$= 2\,T\lambda\mu \sin\tfrac{1}{2} <^{\lambda}_{\mu} = \sqrt{2\,T^2\lambda\mu(1 - \cos <^{\lambda}_{\mu})},$$

$$= \sqrt{2}\,(T^2\lambda\mu + T\lambda\mu\, S\lambda\mu)^{\frac{1}{2}}.$$

If k be the length of γ, our equation becomes

$$(S\lambda\mu\rho)^2 + (S\lambda\rho T\mu + S\mu\rho T\lambda)^2$$

$$= \sqrt{2}\,k(T\lambda\mu - S\lambda\mu)^{\frac{1}{2}}\,T^{-\frac{1}{2}}\lambda\mu(S\lambda\rho T\mu - S\mu\rho T\lambda + c)$$

$$= \sqrt{2}.kT^{-\frac{1}{2}}\lambda\mu(T\lambda\mu - S\lambda\mu)^{\frac{1}{2}}\{S(\lambda\rho T\mu - \mu\rho T\lambda) + c\}.$$

This is an *elliptic paraboloid;* for, if cut by a plane perpendicular to $V\lambda\mu$, or $U(\lambda T\mu + \mu T\lambda)$, the section is a *parabola.* But, if by a plane perpendicular to $U(\lambda T\mu - \mu T\lambda)$, it is an *ellipse.*

To obtain the *hyperbolic paraboloid,* let

$$c_2 = 0, \quad g = S\lambda\mu.$$

The general cyclic form of the equation of the second degree, in this case, reduces to

$$S\lambda\rho S\mu\rho = S\gamma\rho + c.$$

By a transfer of the origin in the plane of λ and μ, γ may be reduced to a direction perpendicular to that plane; and our equation to the form

$$S\lambda\rho S\mu\rho = nS\lambda\mu\rho + c.$$

Cut this by a plane perpendicular to $V\lambda\mu$, that is, by the plane

$$S\lambda\mu\rho = c',$$

and we have

$$S\lambda\rho S\mu\rho = c'',$$

which is an *hyperbola*. If now the surface be cut by planes perpendicular to one of the other axes $(\lambda + \mu)$ or $(\lambda - \mu)$, the planes

$$S\rho(\lambda + \mu) = 0$$

or

$$S\rho(\lambda - \mu) = 0$$

our equation gives

$$\pm S^2\lambda\rho = S\lambda\mu\rho + c.$$

And these are both *parabolas*. These equations for paraboloids can be verified by substituting in the cyclic forms the value of λ and μ in terms of c_1, c_2, and c_3, α_1, α_2, and α_3, given in Hamilton's Elements of Quaternions, § 357, XX., and seq.

In the general equation in δ, for finding the centre

$$(g - S\lambda\mu)\delta + \lambda S\mu\delta + \mu S\lambda\delta = \gamma,$$

if c_1 and c_2 both vanish, and

$$\gamma = di + hk,$$

we have

$$- T\lambda\mu = S\lambda\mu;$$

whence

$$1 = \cos \angle{}^{\lambda}_{\mu}$$

λ is then parallel to μ, and therefore

$$2\lambda S\lambda\delta = di + hk.$$

The centre must be at a finite distance in the direction of λ, which is the same as that of i, at an infinite distance in the direction of k, and indeterminate in that of j; since we may add xj to δ without affecting the equation. The surface must be a *parabolic cylinder*. Now, if h also vanish, k will be in the same condition as j, and δ will be anywhere in a certain plane perpendicular to i. Let us study this case a moment, for it is the simplest of the non-central quadrics under the cyclic form. The general cyclic equation of the second degree

$$(g - S\lambda\mu)\rho^2 + 2\, S\lambda\rho S\mu\rho = 2\, S\gamma\rho + c,$$

in this case, assumes the form

$$S^2\lambda\rho = hS\lambda\rho + c.$$

But this can be reduced to the form

$$S^2\lambda\rho - S\lambda\beta S\lambda\rho - S\lambda\delta S\lambda\rho + S\lambda\beta S\lambda\delta = 0,$$

for we only assume

$$h = S\lambda\beta + S\lambda\delta$$

and

$$c = - S\lambda\beta S\lambda\delta,$$

and these are but two conditions to determine two unknown quantities. Our equation now becomes

$$(S\lambda\rho - S\lambda\beta)(S\lambda\rho - S\lambda\delta) = 0,$$

and may be decomposed into

$$S\lambda(\rho - \beta) = 0$$

and

$$S\lambda(\rho - \delta) = 0.$$

Two parallel planes perpendicular to λ.

Circular Sections.

Almost the only things worthy of notice about the paraboloids are their planes of circular sections; and these are interesting chiefly on account of their connection with the planes of circular sections of the central quadrics.

Differentiating the equation of the paraboloid

$$c_2S^2\alpha_2\rho + c_3S^2\alpha_3\rho = 2kS\alpha_1\rho,$$

we obtain that of its tangent plane

$$2\,c_2S\alpha_2\rho S\alpha_2\rho' + 2\,c_3S\alpha_3\rho S\alpha_3\rho' = 2kS\alpha_1\rho'.$$

At the origin, ρ vanishes, and dropping the accent of ρ', we get for the tangent plane

$$S\alpha_1\rho = 0.$$

Now if we find a sphere with the same tangent plane, obtain the equation of the surface of the second order in which lies the intersection of this sphere with the paraboloid, and finally discover the condition that this surface should degenerate into a pair of planes, we shall have the planes whose intersection with the sphere, and consequently with the paraboloid, will be circles.

Such a sphere will be represented by the equation

$$\rho^2 = 2n S\alpha_1\rho,$$

for this has

$$S\alpha_1\rho = 0$$

for a tangent plane at the origin. Its intersection with the paraboloid is given by

$$c_2 S^2\alpha_2\rho + c_3 S^2\alpha_3\rho - \frac{k}{n}\rho^2 = 0;$$

a cone referred to its vertex. The cone will represent a pair of planes, if

$$c_2 = \frac{-k}{n};$$

for then

$$c_2(S^2\alpha_2\rho + \rho^2) + c_3 S^2\alpha_3\rho = 0.$$

Now if we reduce to Cartesian co-ordinates by the substitution

$$\rho = x\alpha_1 + y\alpha_2 + z\alpha_3,$$

we find that

$$c_2(y^2 - x^2 - y^2 - z^2) + c_3 z^2 = 0,$$

$$-c_2(x^2 + z^2) = -c_3 z^2;$$

and finally

$$c_2 x^2 = (c_3 - c_2)z^2.$$

Since we can always take $c_2 < c_3$, — whatever these roots may be, — the equation just found will be imaginary if c_2 and c_3 have opposite signs. The geometrical interpretation of which is that an hyperbolic paraboloid can have no circular section: a fact almost self-evident.

Let us substitute in the equation

$$c_2 x^2 = (c_3 - c_2)z^2,$$

the cyclic values of c_1, c_2, and c_3 (Ham. Elem., § 357, XXI.),

$$c_2 - c_1 = T\lambda\mu + S\lambda\mu$$

$$c_3 - c_2 = T\lambda\mu - S\lambda\mu.$$

In the present case, $c_1 = 0$; and therefore

$$x^2 = \frac{T\lambda\mu - S\lambda\mu}{T\lambda\mu + S\lambda\mu} z^2 = \frac{1 - \cos \angle\frac{\lambda}{\mu}}{1 + \cos \angle\frac{\lambda}{\mu}} z^2 = z^2 \tan^2 \tfrac{1}{2} \angle\frac{\lambda}{\mu}$$

$$= z^2 \cot^2 \left(\tfrac{1}{2} \circlearrowleft + \tfrac{1}{2} \angle \frac{\lambda}{\mu}\right).$$

Now this represents two lines perpendicular to λ and μ

$$x = \pm z \cot \left(\tfrac{1}{2} \circlearrowright + \tfrac{1}{2} <^{\lambda}_{\mu}\right),$$

or (*vide* Fig. 3, page 246)

$$x = z \cot \left(\tfrac{1}{2} \circlearrowright + <^{\lambda}_{i}\right),$$

and

$$x = z \cot \left(\tfrac{1}{2} \circlearrowright + <^{\lambda}_{k}\right).$$

Introducing j with an indeterminate co-efficient, we have two planes perpendicular to λ and μ, cutting the surface in circular sections. And this result we should be led to expect from the fact that the *paraboloid* is a limiting case between the *ellipsoid* and the *hyperboloid.*

This one example is sufficient to show that, with the property of self-conjugation, the general equation of the second degree loses that simplicity of expression which makes quaternions so singularly applicable to the central quadrics. The very form of the self-conjugate function exhibits some of the fundamental properties of these central surfaces. The *rectangular* transformation depends on the principal axes; the *cyclic*, on the relations of the cyclic normals; while the *focal* shows at once the properties of the focal lines of the asymptotic cone. Other transformations could doubtless be discovered, embodying other well-known properties of these remarkable surfaces.

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